Phys. 460a/506a

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Problem Set 8

Due Wed., Nov. 19

Zero credit will be given for solutions in Mathematica.

1. Find the Green's functions for (weight factors are $\rho = 1$ in this question):

(i)

$$D = -\frac{d^2}{dx^2} - 1, \quad x \in [0,1] \quad u(0) = 0, \quad u'(1) = 0; \tag{1}$$

(ii)

$$D = -\frac{d^2}{dx^2} + 1, \quad x \in (-\infty, \infty). \tag{2}$$

In (ii), you should require $G \to 0$ as x or $x' \to \pm\infty$.

2. Consider first the operator $D = -\frac{d^2}{dx^2}$ on $L^2(0,\infty)$ with domain $\operatorname{dom} D$ the space of functions u such that $||u||$ and $||Du||$ are finite, and that obey the boundary condition $u(0) = 0$. Show that D is self-adjoint. (In quantum mechanics this is the Hamiltonian for a free particle that is reflected off a wall at $x = 0$.) You should say something about the boundary terms at both 0 and ∞.

Now consider instead the more general boundary condition $u(0) + \alpha u'(0) = 0$ (α is a real constant), and show that this also gives a self-adjoint operator. Show that for some values of α there is a (normalizable) eigenfunction obeying the boundary condition, find its eigenvalue, and determine for which values of α this is true. Attempt to interpet these results physically in quantum mechanics.

3. Given that

$$P_1(x) = x \quad \text{and} \quad Q_0 = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \tag{3}$$

are solutions of Legendre's equation corresponding to different eigenvalues,

a) evaluate their overlap integral $\int_{-1}^{1} P_1(x) Q_0(x) dx$, and

b) explain why it is not zero. You should find why the proof of orthogonality of eigenfunctions of distinct eigenvalues for a Hermitian operator apparently does not apply.

4. Consider again Chebyshev's equation

$$\widetilde{D}y + n^2 y \equiv (1-x^2)y'' - xy' + n^2 y = 0. \tag{4}$$

(i) Show that by multiplying by $(1-x^2)^{-1/2}$, we obtain the Sturm-Liouville form with $p(x) = (1-x^2)^{1/2}$, $q(x) = 0$, and weight factor $\rho(x) = (1-x^2)^{-1/2}$ on the interval $x \in [-1, 1]$ (we will view $-n^2$ as the eigenvalue of the above operator $\widetilde{D}$ or of $D = \rho(x)\widetilde{D}$). That is, $\widetilde{D}$ is formally self-adjoint. Note that this S-L problem is singular.

(ii) Find the *leading* behavior of two linearly-independent solutions to Chebyshev's equation as $x \to 1$ (the behavior near $x \to -1$ is similar). Hint: are the singular points $x = \pm 1$ regular singular points in the sense that we discussed for ODEs generally?

Hence show that the endpoints $x = \pm 1$ are both in the limit-circle, not the limit-point case (they will be the same by symmetry).

(iii) We now know that there is a two-parameter family of self-adjoint boundary conditions for the "Chebyshev operator" $\widetilde{D}$, with one parameter for each endpoint. Write down these general boundary conditions for the two endpoints.

Look back at your series solutions of Chebyshev's equation in powers of x that you found in problem set 5, no. 2. Show that for certain values of n^2 there is a polynomial solution, i.e. the series terminates. (These are the Chebyshev polynomials.) Show that these solutions satisfy a particular boundary condition at $x = 1$ and at $x = -1$, which is in the two-parameter family of possibilities. Deduce that the Chebsyhev polynomials form an orthogonal set. (They are actually complete in $L^2(-1, 1|\rho)$ also.)

5. Consider the space $L^2(0, 1)$ of square-integrable complex functions of x on the interval $[0, 1]$, with the standard inner product $\langle u|v\rangle = \int_0^1 \overline{u(x)}v(x)\,dx$, and the first-order differential operator D defined by

$$D = -i\frac{d}{dx} - a, \tag{5}$$

where a is a constant complex number, and domain $\operatorname{dom} D$ the space of functions u in $L^2(0, 1)$ that are differentiable, and obey the periodic boundary condition $u(0) = u(1)$. We can view D as acting on the space of periodic functions.

a) Show that if a is real, then D is self-adjoint, i.e. D is Hermitian, $\langle u|Dv\rangle = \langle Du|v\rangle$, and the same boundary conditions are used for both u and v to verify this (and nothing weaker will work in general). Also, show that D is not Hermitian if a is not real.

b) Show that the Green function for D, the solution of $D_x G(x, x') = \delta(x - x')$, can be written as the eigenfunction expansion

$$G(x, x') = \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n(x-x')}}{2\pi n - a} \tag{6}$$

(where n runs over the set of integers), except when a lies in a certain set of values. What is this set of values of a for which this G does not exist?

c) Note again that D is first order. What sort of non-analytic behavior must G have at $x = x'$ (modulo 1)?

Show that the Green function G can be found explicitly, as follows. Without loss of generality, let us set $x' = 0$ (values elsewhere can be found by translating both x and x'). Show directly from the differential equation for G that it can be written as

$$G(x, 0) = \frac{e^{iax}}{A} \quad (0 < x < 1) \tag{7}$$

(and repeats periodically for other values of x), and determine the value of A. For what values of a does this construction fail? Why?

d) Calculate the Fourier series of the solution for G in part c), and so recover the same result as in part b) by a different method.

6. Show directly that a symmetric Green's function $G(x, x')$ for the one-dimensional version of the Laplace operator,

$$-\frac{d^2 G(x, x')}{dx^2} = \delta(x - x'), \tag{8}$$

is, at least formally,

$$G(x, x') = -|x - x'|/2. \tag{9}$$

Show that the Green's function for the Laplace operator $-\nabla^2$ behaves poorly at infinity, in the sense that it is not in $L^2(\mathbf{R}^d)$ as functions of $\mathbf{x}$ for fixed $\mathbf{x}'$, for dimensions $d = 1, 2, 3$. That is,

$$\int d^d x |G(\mathbf{x}, \mathbf{x}')|^2 = \infty \tag{10}$$

(diverging at large r) for $d = 1, 2, 3$. What happens for $d > 4$ (note that $G(\mathbf{x}, \mathbf{x}') \propto |\mathbf{x} - \mathbf{x}'|^{-(d-2)}$ for $d > 2$)?